Rutgers University: Algebra Written Qualifying Exam

August 2015: Problem 2 Solution

Exercise. Recall that the *algebraic multiplicity* of an eigenvalue of a square matrix is defined as its multiplicity as a root of the characteristic polynomial of that matrix. If A is a square matrix with complex entries, let $exp(A)$ denote the exponential of A, defined as the power series

$$exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{1}{2} A^2 + \dots.$$

Assume all eigenvalues of A are real. If λ is an eigenvalue for A with algebraic multiplicity μ, show that e^λ is an eigenvalue for $exp(A)$, and has the same algebraic multiplicity μ.

Solution.

If $\lambda \in \mathbb{R}$ is an eigenvalue of A then $\exists \vec{v}$ s.t. $A\vec{v} = \lambda \vec{v}$.

$$\begin{aligned} \implies exp(A)\vec{v} &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n \right) \vec{v} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} A^n \vec{v} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \vec{v} \\ &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \right) \vec{v} \\ &= e^\lambda \vec{v} \end{aligned}$$

Thus, if is an eigenvalue for A, then e^λ is an eigenvalue for A.

Let $A = PJP^{-1}$ be the Jordan canonical form of A.

Then $A \sim J$ so they have the same eigenvalues and the same algebraic multiplicity.

$J^n_{\lambda_i, m_i}$ is the upper triangular matrix with diagonal λ_i^n.

$\implies \sum_{n=0}^{\infty} \frac{1}{n!} J^n_{\lambda_i, m_i}$ is an upper triangular matrix with diagonals $\sum_{n=0}^{\infty} \frac{1}{n!} \lambda_i^n = e^{\lambda_i}$

$\implies \sum_{n=0}^{\infty} J_{\lambda_i, m_i} \sim A_i$ has eigenvalues e^{λ_i} with multiplicity m_i

$\implies e^\lambda$ has the same algebraic multiplicity as λ in A